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COLORING ABELIAN GROUPS

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A sufficient (and necessary, if $n=2$) condition for the existence of a particular kind of n-coloring of an abelian group is given, and applied to show that (a) the real line is colorable with two colors so that the distance 1 is forbidden for one color, and the distance $s>0$ for the other, or so that both 1 and s are forbidden for both colors, if and only if s is not the ratio of an odd and an even integer; (b) the chromatic number of Q^2 and Q^3 is 2, but that of Q^n is greater than 2 for $n>3$.

The object of this paper is mainly to give and apply a necessary and sufficient condition for the two-colorability of certain graphs associated with abelian groups. This condition is, of course, equivalent to the condition of non-existence of odd cycles in the graph, but has an algebraic character that makes it easier to apply in certain situations.

G will be an abelian group. An n-coloring of G will be a partition of G into n sets.

Definition. Suppose $h\in G$; $h\neq 0$. An n-coloring of G is *h-alternating* if and only if, for each $g\in G$, the group elements $g+mh$, $0\leq m\leq n-1$, are all colored differently.

A subset $B\subseteq G$ is *h-forbidden* if and only if, for each $g\in G$, $g\in B$ implies $g+h\notin B$. A coloring of G is h-forbidden if and only if each of the sets of the partition is h-forbidden.

If $S\subseteq G$, a coloring of G is *S-alternating*, or *S-forbidden*, if and only if it is h-alternating, respectively h-forbidden, for each $h\in S$.

A subset $S\subseteq G$ is *weakly n-free* if and only if $\sum_{h\in S} m_h h=0$ implies $\sum_{h\in S} m_h\equiv 0 \pmod n$. [Here, and later, at most finitely many of the integers m_h are non-zero. The word 'weakly' is used because the sum of the m_h is required to be divisible by n, not each m_h separately.]

It will be useful to note that if a coloring is h-alternating, then it is h-forbidden; if a set $B\subseteq G$ is h-forbidden, then it is $-h$-forbidden; if an n-coloring is h-alternating, then $g+mh$ and $g+kh$ have the same color if and only if $m\equiv k$ $(\text{mod}\ n)$. In the case $n=2$, this last observation may be extended: if $S\subseteq G$, then for any S-alternating two-coloring, g and $g+\sum_{h\in S} m_h h$ have the same color if and only if $\sum_{h\in S} m_h$ is even.

The case $n=2$ is also distinguished by the following.

Proposition. *A two-coloring B, R of G is $\{h, z\}$-alternating if and only if B is h-forbidden and R is z-forbidden.*

Proof. The 'only if' assertion is trivial. Suppose B is h-forbidden and R is z-forbidden. If g, $g+h\in R$, then $g+z$, $g+z+h\in B$; but this contradicts the assumption that B is h-forbidden. Consequently, at least one of g, $g+h$ is in B, and thus exactly one, since B is h- (and therefore $-h$-) forbidden.

Thus B, R is h-alternating, and, likewise, z-alternating.

[*Remark.* This proposition is a slight improvement over the easy observation that a two-coloring is S-forbidden if and only if it is S-alternating.]

Theorem. *Suppose $S\subseteq G$, and $n>1$. If S is weakly n-free, then there is an S-alternating n-coloring of G. If $n=2$, or if S is a singleton, the converse holds; the existence of an S-alternating n-coloring implies that S is weakly n-free.*

Proof. Suppose S is weakly n-free. Let H denote the subgroup generated by S. Color H by letting

$$B_k=\left\{\sum_{h\in S} m_h h \;\middle|\; \sum_{h\in S} m_h \equiv k \pmod{n}\right\},\quad k=0,\ldots,n-1.$$

The B_k are mutually disjoint by the assumption that S is weakly n-free. Extend the coloring to all of G by coloring the co-sets of H similarly.

The proof of the converse in the special cases mentioned is easy, by previous remarks.

Corollary 1. *Suppose S_1 and S_2 are non-empty subsets of G. There is a two coloring B; R of G such that B is S_1-forbidden and R is S_2-forbidden if and only if $S_1\cup S_2$ is weakly two-free.*

Proof. This is an easy corollary of the Proposition and the Theorem.

Corollary 2. *Suppose $s>0$. There is a two-coloring of the real line (with red and blue, say), such that no two red points are 1 apart, and no two blue points are s apart, if and only if $s\notin K=\{p/q \mid p, q$ are positive integers and $p+q$ is odd$\}$.*

Proof. The conclusion follows from Corollary 1 and the observation that $\{1, s\}$ is weakly 2-free in the additive group of reals if and only if $s\notin K$.

The chromatic number of a subset of a Euclidean space is the smallest number of colors needed to color the set so that no two points a distance 1 apart are the same color (the distance 1 is forbidden).

Corollary 3. *The chromatic number of Q^2 and Q^3 is* 2, *and of Q^4 is greater than* 2.

Proof. To show that the chromatic number of Q^2 is 2, it suffices to show that $S=\{(x, y)\in Q^2 \mid x^2+y^2=1 \text{ and } x=1 \text{ or } y>0\}$ is weakly 2-free (in the additive group). This is essentially a consequence of the fact that if (a, b, c) is a primitive Pythagorean triple, then one of a, b is odd, the other even, and c is odd. Thus, if $\{(1,0), (0,1), (a_1/c_1, b_1/c_1), \ldots, (a_k/c_k, b_k/c_k)\}$ is a finite subset of S (into which it doesn't hurt to throw $(1,0)$ and $(0,1)$), with (a_j, b_j, c_j) being relatively prime Pythagorean triples, and

$$n(1,0)+r(0,1)+\sum_j m_j(a_j/c_j, b_j/c_j)=(0,0),$$

then, multiplying by c, the product of the c_j, an odd number, we have

$$n+\sum m_ja_j\equiv 0 \pmod 2 \quad \text{and} \quad r+\sum m_jb_j\equiv 0 \pmod 2,$$

so

$$n+r+\sum m_j(a_j+b_j)\equiv n+r+\sum m_j\equiv 0 \pmod 2,$$

which was to be shown.

The proof for Q^3 is similar. The point there is that in any primitive Pythagorean quadruple (a, b, c, d), d and exactly one of a, b, c must be odd.

In Q^4, we have $3\cdot[\frac{1}{6}(1,1,3,5)]-1\cdot[\frac{1}{2}(1,1,1,1)]-(0,0,1,0)-2(0,0,0,1)=(0,0,0,0)$, while $3-1-1-2$ is odd, which shows that the unit sphere in Q^4 is not weakly 2-free, which means that Q^4 cannot be colored with two colors so that the distance 1 is forbidden.

Remarks. (i) In Corollary 2, if $s=p/q$, p, q odd positive integers, then the coloring called for could be obtained by alternating colors on the intervals

$$\left[\frac{n}{q}, \frac{n+1}{q}\right), \quad n=0, \pm 1, \ldots.$$

However, if s is irrational, then for every 2-coloring forbidding 1 and s, both colors are dense on the line. To see this, note that $A=\{m+ns \mid m, n \text{ integers}\}$ is dense on the line, and the sets $2A$, $2A+1$ are each monochromatic, of different colors.

(ii) Again regarding Corollary 2, the theorem really says that the line can be 2-colored forbidding all distances in a set $S\subseteq(0,\infty)$ if and only if S is weakly 2-free. The existence of weakly 2-free sets and Zorn's Lemma imply the existence of maximal weakly 2-free sets; it is easy to see that any such set must span the reals, considered as a vector space over Q, and thus must have cardinality $2^{\aleph_0}$. It is generally well known that $2^{\aleph_0}$ distances are forbiddable in a 2-coloring of the line; the power of the theorem in this context is not in the rederivation of this result, but in that the theorem describes which sets of distances are forbiddable.

As a possible application, note that if a weakly 2-free set $S \subseteq (0, \infty)$ could be found with the property that $x^2 + y^2 = 1$, x, $y \geqslant 0$ implies $x \in S$ or $y \in S$, then we could 4-color the plane so that the distance 1 is forbidden for all colors; just color the line with r and b so that all distances in S are forbidden, and then color the plane with the four colors (r, r), (r, b), (b, r), (b, b) in the obvious way. Thus the existence of such a set S would imply that the chromatic number of the plane is 4 (see [2]), although its non-existence would not imply the contrary. However, the author does not know whether or not such an S exists.

(iii) The result that the chromatic number of Q^2 is 2 is known, and a brief proof is given in [2, Theorem 1].

(iv) Consider the following problem scheme: you are given a subset A of a Euclidean space, and k sets of distances $S_1, \ldots, S_k$, and are asked to color A with k colors $c_1, \ldots, c_k$ so that all distances in S_i are forbidden for the color c_i, $i = 1, \ldots, k$. The special cases in which the S_i are singletons are of particular interest. The general problem scheme can be formulated in the terminology of [1], but doesn't seem to be a Euclidean Ramsey problem scheme in any obvious way.

Woodall gives a proof of a theorem of Raiskii [2, Theorems 2 and 3] which certainly implies that if A is Euclidean n-space, and $k = n+1$, $n > 1$, then the problem has no solution for any non-empty $S_1, \ldots, S_k$. On the other hand, D. Greenwell, in unpublished work, has shown that if $k = 3$ and A is the line, then the problem can be solved for every sequence S_1, S_2, S_3 of singletons (and it is a consequence of Corollary 2 that 3 is the smallest k of which this can be said). The Proposition of this paper implies that if A is an additive subgroup of a Euclidean space, then, for $k = 2$, the problem can be solved for S_1, S_2 if and only if a 2-coloring can be found for which all the distances in $S_1 \cup S_2$ are forbidden for both colors; the theorem implies that such a coloring can be found if and only if $\{x \in A \mid |x| \in S_1 \cup S_2\}$ is weakly 2-free (here $|\cdot|$ stands for the Euclidean norm).

Specific unsolved problems of the type described swarm by the gadzillions, but at least two seem to be indicated by the results cited:

(a) In the case when A is the line and $k = 3$, extend Greenwell's result away from singletons, and the consequences of this paper away from $k = 2$, by finding simple conditions on sets S_1, S_2, S_3 of distances for the solvability of the problem.

(b) When A is the plane, find the smallest k, if any exists, such that the problem can be solved for any sequence $S_1, \ldots, S_k$ of singletons.

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References

[1] Erdös, Graham, Montgomery, Rothschild, Spencer and Straus, Euclidean Ramsey Theorems I, J. Combin. Theory (A) 14 (1973) 341–363.
[2] D.R. Woodall, Distances realized by sets covering the plane, J. Combin. Theory (A) 14 (1973) 187–200.